

On Some Classes of $\mathbb{Z}_2\mathbb{Z}_4$ –Linear Codes and their Covering Radius

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Abstract

In this paper we define $\mathbb{Z}_2\mathbb{Z}_4$ –Simplex and MacDonald Codes of type α and β and we give the covering radius of these codes.

Keywords Simplex codes, MacDonald codes, Covering radius, Gray map.

1 Introduction

Much research has concerned the construction of the additive codes especially, Delsarte in 1973, who put the first definition of this codes [12]. For more information concerning the $\mathbb{Z}_2\mathbb{Z}_4$ -additive code the reader is invited to consult (see [3], [4], [5], [6] and [16]). The simplex and MacDonald codes over a finite fields and a finite rings are studied in several version of those articles. (see [1], [7], [9] and [14]). We will use these codes to define a new family of simplex and MacDonald codes is a concatenation of a binary and quaternary simplex and MacDonald codes.

The subject of this paper is the construction of simplex and MacDonald codes over $\mathbb{Z}_2\mathbb{Z}_4$ of type α and β is also their covering radius. The paper is organized as follows. In Section 2, we recall some properties related to $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes. In Section 3, we calculated the covering radius of $\mathbb{Z}_2\mathbb{Z}_4$ - repetition codes. In Section 4 and 5, we describe the construction of $\mathbb{Z}_2\mathbb{Z}_4$ - simplex, MacDonald codes and we estimated theirs covering radius. In Section 6, We concluded a relation between the first order of Reed Muller codes and simplex codes over $\mathbb{Z}_2\mathbb{Z}_4$. Finally in section 7, we give the binary version of these codes.

2 Preliminaries

In this section based on some articles (see [3], [4] and [6]) to select General preliminaries which serve this search.

Denote by \mathbb{Z}_2 and \mathbb{Z}_4 the rings of integers modulo 2 and modulo 4, respectively. Let \mathbb{Z}_2^n and \mathbb{Z}_4^n denote the space of n –tuples over these rings. We say that a binary code is any

nonempty subset C of \mathbb{Z}_2^n , and if that subcode is a vector space then we say that it is a linear code. Similarly, any nonempty subset C of \mathbb{Z}_4^n is called a linear quaternary code.

The codes considered here are subgroups of the space $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. A code C is $\mathbb{Z}_2\mathbb{Z}_4$ -additive if the set of coordinates can be partitioned into two subsets X and Y such that the punctured code of C by deleting the coordinates outside X (respectively, Y) is a binary linear code (respectively, a quaternary linear code). Let C_b be the subcode of C which contains all order two codewords and let k be the dimension of $(C_b)_X$, which is a binary linear code. For the case $\gamma = 0$, we will write $k = 0$. Considering all these parameters, we will say that C (or equivalently $C = \Phi(C)$) is of type $(\gamma, \delta; \lambda, \mu; k)$. The structure of these codes is given as follows. We write $v = (v_1, v_2) \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ where $v_1 = (x_1, \dots, x_\gamma) \in \mathbb{Z}_2^\gamma$ and $v_2 = (y_1, \dots, y_\delta) \in \mathbb{Z}_4^\delta$.

The code C is a subgroup of $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, as such it is isomorphic to $\mathbb{Z}_2^\lambda \times \mathbb{Z}_4^\mu$ for some λ and μ . We say that C is of type $2^\lambda 4^\mu$ as a group. It follows that it has $|C| = 2^{\lambda+2\mu}$ codewords and the number of order two codewords in C is $2^{\lambda+\mu}$. We will take an extension of the usual Gray map $\Phi : \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta \rightarrow \mathbb{Z}_2^n$, where $n = \gamma + 2\delta$ given by

$$\Phi(u, v) = (u, \phi(v_1), \dots, \phi(v_\delta)), \forall u \in \mathbb{Z}_2^\gamma, \forall (v_1, \dots, v_\delta) \in \mathbb{Z}_4^\delta$$

where $\phi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2^2$ is the usual Gray map that is $\phi(0) = (0, 0), \phi(1) = (0, 1), \phi(2) = (1, 1), \phi(3) = (1, 0)$. This Gray map is an isometry which transforms Lee distances defined in $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ to Hamming distances defined in \mathbb{Z}_2^n , where $n = \gamma + 2\delta$.

Let $v_1 \in \mathbb{Z}_2^\gamma$ and $v_2 \in \mathbb{Z}_4^\delta$. Denote by $wt_H(v_1)$ the Hamming weight of v_1 and $wt_L(v_2)$ the Lee weight of v_2 . For a vector $v = (v_1, v_2) \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$, define the weight of v , denoted by $wt(v)$, as $wt_H(v_1) + wt_L(v_2)$, or equivalently, the Hamming weight of $\Phi(v)$. The Euclidean weight is given by the relation $wt(v) = wt_H(v_1) + wt_E(v_2)$, where the Euclidean weight $w_E(u)$ of a vector u is $\sum_{i=1}^n \min \{u_i, (4 - u_i)^2\}$.

Definition 2.1 Let C be a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code, which is a sub group of $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. We say that the binary image $\Phi(C)$ is a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code of binary length $n = \gamma + 2\delta$ and type $(\gamma, \delta, \lambda, \mu; k)$, where λ, μ and k are defined as above.

Let C be a $\mathbb{Z}_2\mathbb{Z}_4$ -additive code. A $\mathbb{Z}_2\mathbb{Z}_4$ -additive code C is not a free module, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\lambda} \lambda_i u_i + \sum_{j=1}^{\mu} \mu_j v_j$$

where, $\lambda_i \in \mathbb{Z}_2$, $\mu_j \in \mathbb{Z}_4$ and for $1 \leq i \leq \lambda$ and $1 \leq j \leq \mu$, $u_i, v_j \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ of order two and order four, respectively. Vectors u_i, v_j gives as a generator matrix G of size $(\lambda + \mu) \times (\gamma + \delta)$ for the code C . Moreover, we can write G as

$$G = \left[\begin{array}{c|c} B_1 & 2B_3 \\ \hline B_2 & Q \end{array} \right]$$

where B_1, B_2 are matrices over \mathbb{Z} of size $\lambda \times \gamma$ and $\mu \times \gamma$, respectively; B_3 is a matrix over \mathbb{Z}_4 of size $\lambda \times \delta$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and Q is a matrix over \mathbb{Z}_4 of size $\mu \times \delta$ with quaternary row vectors of order four.

In [5], it is shown that a $\mathbb{Z}_2\mathbb{Z}_4$ - additive code is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$ - additive code with standard generator matrix of the form

$$G_S = \left[\begin{array}{cc|ccc} I_k & T' & 2T_1 & 0 & 0 \\ 0 & 0 & T_2 & 2I_{\lambda-k} & 0 \\ \hline 0 & S' & S & R & I_\lambda \end{array} \right] \quad (1)$$

where T', T_1, T_2, R, S' are a matrix over \mathbb{Z}_2 and S is a matrix over \mathbb{Z}_4 .

We define the inner product of vector $u, v \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ as

$$\langle u, v \rangle_{\mathbb{Z}_2\mathbb{Z}_4} = 2 \left(\sum_{i=1}^{\gamma} u_i v_i \right) + \sum_{j=\gamma+1}^{\gamma+\delta} u_j v_j \in \mathbb{Z}_4$$

The $\mathbb{Z}_2\mathbb{Z}_4$ - additive dual code of C , denoted by C^\perp , is defined in the standard way

$$C^\perp = \{v \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta \mid \langle u, v \rangle_{\mathbb{Z}_2\mathbb{Z}_4} = 0 \text{ for all } u \in C\}.$$

We will also call C^\perp the additive dual code of C . We use the canonical generator matrix as in 1 to construct a canonical generator matrix of C^\perp , to get a parity-check matrix

$$H_S = \left[\begin{array}{cc|ccc} T' & I_{\gamma-k} & 0 & 0 & 2S'^t \\ 0 & 0 & 0 & I_{\lambda-k} & 2R^t \\ \hline T_1^t & 0 & I_{\delta+k-\lambda-\mu} & T_2^t & -(S + RT_2)^t \end{array} \right] \quad (1)$$

where T', T_1, T_2, R, S' are a matrix over \mathbb{Z}_2 and S is a matrix over \mathbb{Z}_4 .

2.1 Covering Radius of Codes

In this section, we introduce the basic notions of the covering radius of codes over $\mathbb{Z}_2\mathbb{Z}_4$. The covering radius of a code C , denoted $r(C)$, is the smallest number r such that the spheres covering radius of radius r around the codewords of C cover the sets $(\mathbb{Z}_2 \times \mathbb{Z}_4)^n$. The covering radius of a code C over $\mathbb{Z}_2\mathbb{Z}_4$ with respect to the Lee and Eucliden distances is given by

$$r_L(C) = \max_{u \in (\mathbb{Z}_2 \times \mathbb{Z}_4)^n} \{\min_{c \in C} d_L(u, c)\}$$

and

$$r_E(C) = \max_{u \in (\mathbb{Z}_2 \times \mathbb{Z}_4)^n} \{ \min_{c \in C} d_E(u, c) \}$$

respectively. It is easy to see that $r_L(C)$, $r_E(C)$ are the minimum values r_L , r_E such that

$$(\mathbb{Z}_2 \times \mathbb{Z}_4)^n = \cup_{c \in C} S_{r_L}(c) \text{ and } (\mathbb{Z}_2 \times \mathbb{Z}_4)^n = \cup_{c \in C} S_{r_E}(c)$$

respectively, where

$$S_{r_L}(u) = \{v \in (\mathbb{Z}_2 \times \mathbb{Z}_4)^n \mid d(u, v) \leq r_L\}$$

and

$$S_{r_E}(u) = \{v \in (\mathbb{Z}_2 \times \mathbb{Z}_4)^n \mid d(u, v) \leq r_E\}$$

for any element $u \in (\mathbb{Z}_2 \times \mathbb{Z}_4)^n$.

Proposition 2.2 *Let C be a code over $(\mathbb{Z}_2 \times \mathbb{Z}_4)^n$ and $\Phi(C)$ the Gray map images of C . Then $r_L(C) = r(\Phi(C))$.*

The lower and upper bounds on the covering radius of codes over $\mathbb{Z}_2\mathbb{Z}_4$ is given in several theorems and proposition (see, [13], [17] and [18]).

Proposition 2.3 *For any code C of length n over $\mathbb{Z}_2\mathbb{Z}_4$,*

$$\frac{2^{2n}}{|C|} \leq \sum_{i=0}^{r_L(C)} \binom{2n}{i}$$

$$\frac{2^{2n}}{|C|} \leq \sum_{i=0}^{r_E(C)} (V_i), \text{ where } \sum_{i=0}^{5n} V_i x^i = (1 + 3x + 2x^2 + x^4 + x^5)^n$$

Proof. The proof of inequality over $\mathbb{Z}_2\mathbb{Z}_4$ is similar to the proof over \mathbb{Z}_4 given in [2].

Let C be a code over $\mathbb{Z}_2\mathbb{Z}_4$ and

$$s(C^\perp) = |\{i \mid A_i(C^\perp) \neq 0, i \neq 0\}|$$

where $A_i(C^\perp)$ is the number of codewords of weight i in C^\perp . Delsarte in [11] proved that the covering radius $r(C)$ of C is given by $r(C) \leq s(C^\perp)$. This is known as the Delsarte bound. Before to define the bound of Delsarte we give the following Lemma

Lemma 2.4 [2] *For a code C over $\mathbb{Z}_2\mathbb{Z}_4$, $r_L(C) \leq r_E(C) \leq 3r_L(C)$.*

Theorem 2.5 (Delesarte Bound) [2] *Let C be a code over $\mathbb{Z}_2\mathbb{Z}_4$ then $r_L(C) \leq s(C^\perp)$ and $r_E(C) \leq 3s(C^\perp)$.*

A coset of the code C defined by the vector $u \in (\mathbb{Z}_2 \times \mathbb{Z}_4)^n$ is the set

$$u + C = \{u + v \mid v \in C\}$$

A coset leader of C is a vector in $u + C$ of smallest weight. When the code is linear its covering radius is equal to the weight of the heaviest coset leader. Hence we have the following proposition

Proposition 2.6 [13] *The covering radius of the linear code C is equal to the maximum weight of a coset leader.*

The following result of Mattson is useful for computing covering radii of codes over rings generalized easily from codes over finite fields.

Proposition 2.7 *If C_0 and C_1 are codes over $\mathbb{Z}_2\mathbb{Z}_4$ generated by matrices G_0 and G_1 respectively and if C is the code generated by*

$$G = \left[\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right]$$

then $r_d(C) \leq r_d(C_0) + r_d(C_1)$ and the covering radius of D (concatenation of C_0 and C_1) satisfy the following $r_d(D) \geq r_d(C_0) + r_d(C_1)$ for all distances d over $\mathbb{Z}_2\mathbb{Z}_4$.

3 The Covering Radius of $\mathbb{Z}_2\mathbb{Z}_4$ -Repetition Codes

The repetition code C over a finite field $\mathbb{F}_q = \{\alpha_0, \alpha_1, \dots, \alpha_{q-2}\}$ is an $[n, 1, n]$ -code such as $C = \{(\alpha\alpha\cdots\alpha)/\alpha \in \mathbb{F}_q\}$. The covering radius of every $[n, 1, n]$ -code is $\lceil \left(\frac{q-1}{q}\right)n \rceil$. In particular, this holds for the binary repetition code. In [13], various classes of repetition codes over \mathbb{Z}_4 have been studied and their covering radius has been obtained. Now we generalize those results for codes over $\mathbb{Z}_2\mathbb{Z}_4$. Consider the repetition codes over $\mathbb{Z}_2\mathbb{Z}_4$. One can define seven basic repetition codes C_{α_i} , ($1 \leq i \leq 7$) of length n over $\mathbb{Z}_2\mathbb{Z}_4$ generated by $G_{\alpha_1} = [0101\cdots 01]$, $G_{\alpha_2} = [0202\cdots 02]$, $G_{\alpha_3} = [0303\cdots 03]$, $G_{\alpha_4} = [1010\cdots 10]$, $G_{\alpha_5} = [1111\cdots 11]$, $G_{\alpha_6} = [1212\cdots 12]$, $G_{\alpha_7} = [1313\cdots 13]$. So the repetition codes are $C_{\alpha_1} = C_{\alpha_3} = \{(00\cdots 00), (01\cdots 01), (02\cdots 02), (03\cdots 03)\}$, $C_{\alpha_2} = \{(00\cdots 00), (02\cdots 02)\}$, $C_{\alpha_4} = \{(00\cdots 00), (10\cdots 10)\}$, $C_{\alpha_5} = C_{\alpha_7} = \{(00\cdots 00), (01\cdots 01), (02\cdots 02), (03\cdots 03), (10\cdots 10), (11\cdots 11), (12\cdots 12), (13\cdots 13)\}$, following $C_{\alpha_6} = \{(00\cdots 00), (02\cdots 02), (10\cdots 10), (12\cdots 12)\}$. The theorems determine the covering radius of C_{α_i} for ($1 \leq i \leq 7$).

Theorem 3.1 $r_E(C_{\alpha_1}) = r_E(C_{\alpha_3}) = \frac{3n}{4}$ and $r_L(C_{\alpha_1}) = r_L(C_{\alpha_3}) = \frac{3n}{2}$.

Proof. We know that $r_E(C_{\alpha_i}) = \max_{x \in (\mathbb{Z}_2\mathbb{Z}_4)^n} \{d_E(x, C_{\alpha_i})\}$. Let $x \in (\mathbb{Z}_2\mathbb{Z}_4)^n$. If x has composition of $(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$, where $\sum_{j=0}^7 t_j = n$ then $d_E(x, \overline{00}) = n - t_0 + 3t_2 + t_5 + 4t_6 + t_7$, $d_E(x, \overline{01}) = n - t_1 + 3t_3 + t_6 + 4t_7$, $d_E(x, \overline{02}) = n - t_2 + 4t_0 + t_1 + 3t_4 + t_7$ and $d_E(x, \overline{03}) = n - t_3 + t_0 + 4t_1 + t_2 + 3t_5$. Thus $d_E(x, C_{\alpha_1}) = \min(n - t_0 + 3t_2 + t_5 + 4t_6 + t_7, (n - t_1 + 3t_3 + t_6 + 4t_7), (n - t_2 + 4t_0 + t_1 + 3t_4 + t_7), (n - t_3 + t_0 + 4t_1 + t_2 + 3t_5)) \leq \frac{3n}{4}$. If $x = x_1x_2x_3x_4 = \overbrace{00 \dots 00}^{\frac{n}{4}} \overbrace{01}^{\frac{n}{4}} \overbrace{02}^{\frac{n}{4}} \overbrace{03}^{\frac{n}{4}} \in (\mathbb{Z}_2\mathbb{Z}_4)^n$, then $d_E(x, \overline{00}) = d_E(x, \overline{02}) = d_E(x, \overline{03}) = \frac{n}{8} + 4(\frac{n}{8}) + \frac{n}{8} = \frac{3n}{4}$. Thus $r_E(C_{\alpha_1}) \geq \frac{3n}{4}$. Hence $r_E(C_{\alpha_1}) = r_E(C_{\alpha_3}) = \frac{3n}{4}$. The gray map $\Phi(C_{\alpha_1})$ is a binary repetition code of length $3n$ hence $r_L(C_{\alpha_1}) = r_L(C_{\alpha_3}) = \frac{3n}{2}$. \square

Theorem 3.2 $r_E(C_{\alpha_5}) = r_E(C_{\alpha_7}) = n$ and $r_L(C_{\alpha_5}) = r_L(C_{\alpha_7}) = \frac{3n}{2}$.

Proof.

See the first part of Theorem 3.1 is to that $r_E(C_{\alpha_5}) \leq n$. If $x = \overbrace{00 \dots 00}^{\frac{n}{8}} \overbrace{01}^{\frac{n}{8}} \overbrace{02}^{\frac{n}{8}} \overbrace{03}^{\frac{n}{8}} \overbrace{10}^{\frac{n}{8}} \overbrace{11}^{\frac{n}{8}} \overbrace{12}^{\frac{n}{8}} \overbrace{13}^{\frac{n}{8}} \in (\mathbb{Z}_2\mathbb{Z}_4)^n$, then $d_E(x, \overline{00}) = d_E(x, \overline{01}) = d_E(x, \overline{02}) = d_E(x, \overline{03}) = d_E(x, \overline{10}) = d_E(x, \overline{11}) = d_E(x, \overline{12}) = d_E(x, \overline{13}) = \frac{n}{16} + 4(\frac{n}{16}) + \frac{n}{16} + \frac{n}{16} + \frac{n}{8} + \frac{n}{16} + 4(\frac{n}{16}) + \frac{n}{8} = n$. Thus $r_E(C_{\alpha_5}) \geq n$. Hence $r_E(C_{\alpha_5}) = r_E(C_{\alpha_7}) = n$. \square

Theorem 3.3 $r_E(C_{\alpha_2}) = n, r_E(C_{\alpha_4}) = \frac{n}{4}, r_E(C_{\alpha_6}) = \frac{5n}{4}$ and $r_L(C_{\alpha_2}) = r_L(C_{\alpha_4}) = r_L(C_{\alpha_6}) = \frac{3n}{2}$.

Proof. The proof is similar to proof of Theorem 3.1 and 3.2, hence omitted. \square

In order to determine the covering radius of Simplex code of type α and β over $\mathbb{Z}_2\mathbb{Z}_4$, we have to define a block repetition code over $\mathbb{Z}_2\mathbb{Z}_4$ and find its covering radius. Thus the covering radius of the block repetition code $BRep^n : (n = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7, 2^3, d_L = 6n, d_E = \min\{(n_1 + 4n_2 + n_3 + n_4 + 2n_5 + 5n_6 + 2n_7), (n_1 + 4n_2 + n_3 + n_5 + 4n_6 + n_7), (4n_1 + n_2 + 4n_3 + 4n_5 + n_6 + 4n_7), (n_4 + n_5 + n_6 + n_7), (4n_1 + 4n_3 + n_4 + 5n_5 + n_6 + 5n_7)\})$ generated by

$$G = \left(\overbrace{01 \dots 01}^{n_1} \overbrace{02 \dots 02}^{n_2} \overbrace{03 \dots 03}^{n_3} \overbrace{10 \dots 10}^{n_4} \overbrace{11 \dots 11}^{n_5} \overbrace{12 \dots 12}^{n_6} \overbrace{13 \dots 13}^{n_7} \right)$$

is given in the following theorems.

Theorem 3.4 $r_E(BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}) = \frac{1}{4}[3(n_1 + n_3) + n_4 + 5n_6] + (n_2 + n_5 + n_7)$ and $r_E(BRep_\alpha^{7n}) = 6n$

Proof. By proposition 2.7 and Theorem 3.1, 3.2 and 3.3 we have

$r_E(BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}) \geq \frac{1}{4}[3(n_1+n_3)+n_4+5n_6]+(n_2+n_5+n_7)$. Let $x = x_1x_2x_3x_4x_5x_6x_7 \in (\mathbb{Z}_2\mathbb{Z}_4)^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$ with $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ have compositions of $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7), (b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7), (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7), (d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_7), (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7), (f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7), (g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7)$ such that $n_1 = \sum_{j=0}^7 a_j, n_2 = \sum_{j=0}^7 b_j, n_3 = \sum_{j=0}^7 c_j, n_4 = \sum_{j=0}^7 d_j, n_5 = \sum_{j=0}^7 e_j, n_6 = \sum_{j=0}^7 f_j, n_7 = \sum_{j=0}^7 g_j$.

$d_E(x, \overline{00}) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_0 + 3e_2 + e_5 + 4e_6 + e_7 + n_6 - f_0 + 3f_2 + f_5 + 4f_6 + f_7 + n_7 - g_0 + 3g_2 + g_5 + 4g_6 + g_7$, where $\overline{00} = 0000 \dots 00$ is the first vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_1}) = n_1 - a_1 + 3a_3 + a_6 + 4a_7 + n_2 - b_2 + 4b_0 + b_1 + 3b_4 + b_7 + n_3 - c_3 + c_0 + 4c_1 + c_2 + 3c_5 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_5 + e_2 + 4e_3 + 3e_7 + n_6 - f_6 + 3f_0 + f_3 + 4f_4 + f_5 + n_7 - g_7 + 3g_1 + g_4 + 4g_5 + g_6$, where $\overline{y_1} = \overbrace{01 \dots 0102 \dots 0203 \dots 0310 \dots 1011 \dots 1112 \dots 1213 \dots 13}^{n_1 \dots n_7}$ is the second vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_2}) = n_1 - a_1 + 3a_3 + a_6 + 4a_7 + n_2 - b_2 + 4b_0 + b_1 + 3b_4 + b_7 + n_3 - c_3 + c_0 + 4c_1 + c_2 + 3c_5 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_1 + 3e_3 + e_6 + 4e_7 + n_6 - f_2 + 4f_0 + f_1 + 3f_4 + f_7 + n_7 - g_3 + g_0 + 4g_1 + g_2 + 3g_5$, where $\overline{y_2} = \overbrace{01 \dots 0102 \dots 0203 \dots 0300 \dots 0001 \dots 0102 \dots 0203 \dots 03}^{n_1 \dots n_7}$ is the third vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_3}) = n_1 - a_2 + 4a_0 + a_1 + 3a_4 + a_7 + n_2 - b_1 + 3b_3 + b_6 + 4b_7 + n_3 - c_2 + 4c_0 + c_1 + 3c_4 + c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_2 + 4e_0 + e_1 + 3e_4 + e_7 + n_6 - f_1 + 3f_3 + f_6 + 4f_7 + n_7 - g_2 + 4g_0 + g_1 + 3g_4 + g_7$, where $\overline{y_3} = \overbrace{02 \dots 0201 \dots 0102 \dots 0200 \dots 0002 \dots 0201 \dots 0102 \dots 02}^{n_1 \dots n_7}$ is the fourth vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_4}) = n_1 - a_3 + a_0 + 4a_1 + a_2 + 3a_5 + n_2 - b_2 + 4b_0 + b_1 + 3b_4 + b_7 + n_3 - c_1 + 3c_3 + c_6 + 4c_7 + n_4 - d_0 + 3d_2 + d_5 + 4d_6 + d_7 + n_5 - e_3 + e_0 + 4e_1 + e_2 + 3e_5 + n_6 - f_2 + 4f_0 + f_1 + 3f_4 + f_7 + n_7 - g_1 + 3g_3 + g_6 + 4g_7$, where $\overline{y_4} = \overbrace{03 \dots 0302 \dots 0201 \dots 0100 \dots 0003 \dots 0310 \dots 1001 \dots 01}^{n_1 \dots n_7}$ is the fifth vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_5}) = n_1 - a_0 + 3a_2 + a_5 + 4a_6 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_0 + 3c_2 + c_5 + 4c_6 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_5 - e_4 + e_1 + 4e_2 + e_3 + 3e_6 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_4 + g_1 + 4g_2 + g_3 + 3g_6$, where $\overline{y_5} = \overbrace{00 \dots 0000 \dots 0000 \dots 0010 \dots 1010 \dots 1010 \dots 1010 \dots 10}^{n_1 \dots n_7}$ is the sixth vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_6}) = n_1 - a_2 + 4a_0 + a_1 + 3a_4 + a_7 + n_2 - b_0 + 3b_2 + b_5 + 4b_6 + b_7 + n_3 - c_2 + 4c_0 + c_1 + 3c_4 + c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_6 - e_6 + 3e_0 + e_3 + 4e_4 + e_5 + n_6 - f_4 + f_1 + 4f_2 + f_3 + 3f_6 + n_7 - g_6 + 3g_0 + g_3 + 4g_4 + g_5$, where $\overline{y_6} = \overbrace{02 \dots 0200 \dots 0002 \dots 0210 \dots 1012 \dots 1210 \dots 1012 \dots 12}^{n_1 \dots n_7}$

is the seventh vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$.

$d_E(x, \overline{y_7}) = n_1 - a_3 + a_0 + 4a_1 + a_2 + 3a_5 + n_2 - b_2 + 4b_0 + b_1 + 3b_4 + b_7 + n_3 - c_1 + 3c_3 + c_6 + 4c_7 + n_4 - d_4 + d_1 + 4d_2 + d_3 + 3d_6 + n_7 - e_3 + e_0 + 4e_1 + e_2 + 3e_5 + n_2 - f_2 + 4f_0 + f_1 + 3f_4 + f_7 + n_5 - g_5 + g_2 + 4g_3 + 3g_7$, where $\overline{y_7} = \overbrace{03 \dots 02}^{n_1} \overbrace{302 \dots 02}^{n_2} \overbrace{01 \dots 01}^{n_3} \overbrace{10 \dots 10}^{n_4} \overbrace{13 \dots 13}^{n_5} \overbrace{12 \dots 12}^{n_6} \overbrace{11 \dots 11}^{n_7}$ is the eighth vector of $BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}$. Thus $r_E(BRep^{n_1+n_2+n_3+n_4+n_5+n_6+n_7}) \leq \frac{1}{4} [3(n_1 + n_3) + n_4 + 5n_6] + (n_2 + n_5 + n_7)$.

The code has constant Lee weight $6n$. Thus for $x = 1111 \dots 11 \in (\mathbb{Z}_2\mathbb{Z}_4)^{7n}$, we have $d_L(x, BRep^{7n}) = 6n$. \square

4 $\mathbb{Z}_2\mathbb{Z}_4$ -Simplex Code of Type α and β

In this part, we have going to study the simplex code over $\mathbb{Z}_2 \mathbb{Z}_4$ of type α and β we will discuss the properties of these codes. Type α simplex code S_k^α is linear code over $\mathbb{Z}_2 \mathbb{Z}_4$ with parameters $[2^{3k+1}, 2k, d_L, d_E]$ has a generator matrix which after a suitable permutation of coordinates can be written in the form

$$\Theta_k^\alpha = \left[m_k^\alpha \mid m_k^\alpha \mid \dots \mid m_k^\alpha \mid G_k^\alpha \mid G_k^\alpha \mid \dots \mid G_k^\alpha \right], \text{ for } k \geq 1 \quad (1)$$

where m_k^α is a generator matrix of binary simplex code $S_{2,k}^\alpha$ of type α repeat 2^{2k} times in Θ_k^α and G_k^α is a generator matrix of simplex code $S_{4,k}^\alpha$ over \mathbb{Z}_4 of type α repeat 2^k times in Θ_k^α

Type β simplex code S_k^β is a punctured version of S_k^α with the parameters $[2^{3(k-1)}(2^k - 1), 2k, d_L, d_E]$ and has a generator matrix Θ_k^β of the form

$$\Theta_k^\beta = \left[m_k^\beta \mid m_k^\beta \mid \dots \mid m_k^\beta \mid G_k^\beta \mid \dots \mid G_k^\beta \right], \text{ for } k \geq 3 \quad (2)$$

where m_k^β is a generator matrix of binary simplex code of type β repeat 2^k times in Θ_k^β and G_k^β is a generator matrix of simplex code over \mathbb{Z}_4 of type β repeat 2^{k-1} times in Θ_k^β .

4.1 The Covering Radius of $\mathbb{Z}_2\mathbb{Z}_4$ -Simplex Codes of Type α and β

The following two results are two upper bounds of the covering radius of codes over $\mathbb{Z}_2\mathbb{Z}_4$ with respect to Lee and Euclidean weight.

Theorem 4.1 $r_L(S_k^\alpha) = 2^{3k+1}$ and $r_E(S_k^\alpha) \leq 2^k \cdot \left(\frac{17 \cdot 2^{2k} - 2}{6} \right)$.

Proof. $\mathbb{Z}_2\mathbb{Z}_4$ -Simplex code of type α is of constant Lee weight equal 2^{3k+1} . Hence by definition, $r_L(S_k^\alpha) \geq 2^{3k+1}$. On the other hand by the matrix 1, the result of Maston (see Proposition 2.7) for finite rings and Theorem 3.4, we get

$$\begin{aligned}
r_L(S_k^\alpha) &\leq r_L(2^{2k}S_{2,k}^\alpha) + r_L(2^kS_{4,k}^\alpha) \\
&\leq 2^{2k}r_L(S_{2,k}^\alpha) + 2^kr_L(S_{4,k}^\alpha) \\
&\leq 2^{2k}r_H(S_{2,k}^\alpha) + 2^kr_L(S_{4,k}^\alpha) \\
&\leq 2^{2k}[(2^k + 2^{k-1} + \dots + 2^1) + r_L(S_{2,1}^\alpha)] \\
&\quad + 2^k[(3 \cdot 2^{2(k-1)} + 3 \cdot 2^{2(k-2)} + \dots + 3 \cdot 2^{2 \cdot 1}) + r_L(S_{4,1}^\alpha)] \\
&\leq 2^{2k}[(2^k - 1) + 1] + 2^k[(2^{2k} - 2) + 1] \\
&\leq 2^{2k} \cdot 2^k + 2^k \cdot 2^{2k} \\
&\quad 2^{3k+1}
\end{aligned}$$

Thus $r_L(S_k^\alpha) = 2^{3k+1}$.

Similar arguments can be used to show that (using Theorem 3.4)

$$\begin{aligned}
r_E(S_k^\alpha) &\leq r_E(2^{2k}S_{2,k}^\alpha) + r_E(2^kS_{4,k}^\alpha) \\
&\leq 2^{2k}r_E(S_{2,k}^\alpha) + 2^kr_E(S_{4,k}^\alpha) \\
&\leq 2^{2k}r_H(S_{2,k}^\alpha) + 2^kr_E(S_{4,k}^\alpha) \\
&\leq 2^{2k} \cdot 2^k + 2^k \cdot \frac{11(2^{2k}-1)+9}{6} \\
&\leq 2^k \cdot \left(\frac{17 \cdot 2^{2k} - 2}{6}\right)
\end{aligned}$$

□

Similar arguments will compute the covering radius of $\mathbb{Z}_2\mathbb{Z}_4$ -Simplex code of type β .

Theorem 4.2 *The covering radius of $\mathbb{Z}_2\mathbb{Z}_4$ -Simplex code of type β is given by*

1. $r_L(S_k^\beta) \leq 2^{2k}(2^k - 1) + 2^k(2^{k-1} - 2)$
2. $r_E(S_k^\beta) \leq 2^k\left(\frac{17}{6} \cdot 2^{2k} - 2 \cdot 2^k - \frac{443}{6}\right)$

Proof. By the matrix 2, proposition 2.7 and theorem 3.4, we get

$$\begin{aligned}
r_L(S_k^\beta) &\leq r_L(2^{2k}S_{2,k}^\beta) + r_L(2^kS_{4,k}^\beta) \\
&\leq 2^{2k}r_L(S_{2,k}^\beta) + 2^kr_L(S_{4,k}^\beta) \\
&\leq 2^{2k}r_H(S_{2,k}^\beta) + 2^kr_L(S_{4,k}^\beta) \\
&\leq 2^{2k}(2^{k-1} - 1) + 2^k(2^{k-1}(2^k - 1) - 2) \\
&\quad 2^{2k}(2^k - 1) + 2^k(2^{k-1} - 2)
\end{aligned}$$

Similar arguments can be used to show that (using Theorem 3.4)

$$\begin{aligned}
r_E(S_k^\beta) &\leq r_E(2^{2k}S_{2,k}^\beta) + r_E(2^kS_{4,k}^\beta) \\
&\leq 2^{2k}r_E(S_{2,k}^\beta) + 2^kr_E(S_{4,k}^\beta) \\
&\leq 2^{2k}r_H(S_{2,k}^\beta) + 2^kr_E(S_{4,k}^\beta) \\
&\leq 2^{2k}(2^{k-1} - 1) + 2^k[2^k(2^{k+1} - 1) + \frac{1}{3}(4^k - 1) - \frac{147}{2}] \\
&\leq 2^k\left(\frac{17}{6} \cdot 2^{2k} - 2 \cdot 2^k - \frac{443}{6}\right)
\end{aligned}$$

□

Theorem 4.3 $r_L(S_k^{\alpha^\perp}) = r_L(S_k^{\beta^\perp}) = 1$, $r_E(S_k^{\alpha^\perp}) \leq 3$ and $r_L(S_k^{\beta^\perp}) \leq 3$

Proof. The bound of Delsarte gives, $r_L(S_k^{\alpha^\perp}) \leq 1$ and $r_L(S_k^{\beta^\perp}) \leq 1$. So equality is satisfied. Lemma 2.4 show that $r_E(S_k^{\alpha^\perp}) \leq 3$ and $r_L(S_k^{\beta^\perp}) \leq 3$. □

5 $\mathbb{Z}_2\mathbb{Z}_4$ -MacDonald Code of Type α and β and their Covering Radius

The MacDonald code $\mathcal{M}_{k,u}(q)$ over the finite field \mathbb{F}_q is a unique $\left[\frac{q^k - q^u}{q-1}, k, q^{k-1} - q^{u-1}\right]$ code in which every nonzero codeword has weight either q^{k-1} or $q^{k-1} - q^{u-1}$. Let $1 \leq u \leq k-1$, $m_{k,u}^\alpha$ (resp., $m_{k,u}^\beta$) be the matrix obtained from the matrix of binary simplex code m_k^α (resp., m_k^β) by deleting columns corresponding to the columns of the matrix m_u^α and $0_{2^{2u} \times (k-u)}$ (resp., m_u^β and $0_{2^{2u} \times (k-u)}$). i.e,

$$m_{k,u}^\alpha = \left[m_k^\alpha \quad \setminus \quad \frac{0_{2^{2u} \times (k-u)}}{m_u^\alpha} \right] \quad (3)$$

and

for $k \geq 3$

$$m_{k,u}^\beta = \left[m_k^\beta \quad \setminus \quad \frac{0_{2^{2u} \times (k-u)}}{m_u^\beta} \right] \quad (4)$$

In [9] authors have defined the MacDonald codes over \mathbb{Z}_4 using the generator matrices of simplex codes. For $1 \leq u \leq k-1$, let $G_{k,u}^\alpha$ (resp., $G_{k,u}^\beta$) be the matrix obtained from G_k^α (resp., G_k^β) by deleting columns corresponding to the columns of the matrix G_u^α and $0_{2^{2u} \times (k-u)}$ (resp., G_u^β and $0_{2^{2u} \times (k-u)}$) i.e,

$$G_{k,u}^\alpha = \left[G_k^\alpha \quad \setminus \quad \frac{0_{2^{2u} \times (k-u)}}{G_u^\alpha} \right] \quad (5)$$

and

for $k \geq 3$

$$G_{k,u}^\beta = \left[G_k^\beta \quad \setminus \quad \frac{0_{2^{2u} \times (k-u)}}{G_u^\beta} \right] \quad (6)$$

Now, we will construct the MacDonald codes over $\mathbb{Z}_2\mathbb{Z}_4$ of type α and β by using the generator matrix of the $\mathbb{Z}_2\mathbb{Z}_4$ -simplex codes of type α and β . If $1 \leq u \leq k-1$, let $\Theta_{k,u}^\alpha$ (resp., $\Theta_{k,u}^\beta$) be the matrix of MacDonald codes $\mathcal{M}_{k,u}^\alpha$ (resp., $\mathcal{M}_{k,u}^\beta$) with parameters $[2^{3k+1} - 2^{k+u}(2^k - 2^u)]$ (resp., $[2^{2k-1}(2^{2k-1} + 1)(2^k - 1) - 2^{k+u-1}(2^{2u-3} + 1)(2^u - 1)]$) obtained from Θ_k^α (resp., Θ_k^β) by deleting columns corresponding to the columns of the matrix Θ_u^α and $0_{2^{2u} \times (k-u)}$ (resp., Θ_u^β and $0_{2^{2u} \times (k-u)}$). i.e,

for $k \geq 1$

$$\Theta_{k,u}^\alpha = \left[\begin{array}{c|c|c|c|c|c} m_{k,u}^\alpha & \cdots & m_{k,u}^\alpha & G_{k,u}^\alpha & \cdots & G_{k,u}^\alpha \end{array} \right], \quad (7)$$

where $m_{k,u}^\alpha$ (resp., $G_{k,u}^\alpha$) repeat 2^{2k} (resp., 2^k) times in $\Theta_{k,u}^\alpha$.

and for $k \geq 3$

$$\Theta_{k,u}^\beta = \left[\begin{array}{c|c|c|c|c|c} m_{k,u}^\beta & \cdots & m_{k,u}^\beta & G_{k,u}^\beta & \cdots & G_{k,u}^\beta \end{array} \right], \quad (8)$$

where $m_{k,u}^\beta$ (resp., $G_{k,u}^\beta$) repeat 2^k (resp., 2^{k-1}) times in $\Theta_{k,u}^\beta$.

Next theorems provides basic bounds on the covering radius of MacDonal codes over $\mathbb{Z}_2\mathbb{Z}_4$ of type α .

Theorem 5.1 for $u \leq r \leq k$

1. $r_L(\mathcal{M}_{k,u}^\alpha) \leq [2^{3 \cdot k+1} - 2^{k+r} (2^r + 2^k)] + [2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\alpha,2}) + 2^k r_L(\mathcal{M}_{r,u}^{\alpha,4})]$
2. $r_E(\mathcal{M}_{k,u}^\alpha) \leq \frac{11}{6} [2^{3 \cdot k+2} - 2^{k+r} (2^r + 3 \cdot 2^k)] + [2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\alpha,2}) + 2^k r_E(\mathcal{M}_{r,u}^{\alpha,4})]$

Proof.

$$\begin{aligned} r_L(\mathcal{M}_{k,u}^\alpha) &\leq r_L(2^{2 \cdot k} \mathcal{M}_{k,u}^{\alpha,2}) + r_L(2^k \mathcal{M}_{k,u}^{\alpha,4}) \\ &\leq 2^{2 \cdot k} r_L(\mathcal{M}_{k,u}^{\alpha,2}) + 2^k r_L(\mathcal{M}_{k,u}^{\alpha,4}) \\ &\leq 2^{2 \cdot k} r_H(\mathcal{M}_{k,u}^{\alpha,2}) + 2^k r_L(\mathcal{M}_{k,u}^{\alpha,4}) \\ &\leq 2^k (2^{2 \cdot k} - 2^{2 \cdot r}) + 2^k r_L(\mathcal{M}_{r,u}^{\alpha,4}) + 2^{2 \cdot k} (2^k - 2^r) + 2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\alpha,2}) \\ &\leq [2^{3 \cdot k+1} - 2^{k+r} (2^r + 2^k)] + [2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\alpha,2}) + 2^k r_L(\mathcal{M}_{r,u}^{\alpha,4})] \end{aligned}$$

□

Similar arguments holds for $r_E(\mathcal{M}_{k,u}^\alpha)$. Similarly using the matrix 7, Proposition 2.7 and Theorem 3.4 following bounds can be obtained MacDonal code of type β .

Theorem 5.2 for $u \leq r \leq k$

1. $r_L(\mathcal{M}_{k,u}^\beta) \leq 2^{2 \cdot k-1} (3 \cdot 2^k - 1) - 2^{k+r-1} (2^{k-1} + 2^r - 1) + [2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\beta,2}) + 2^k r_L(\mathcal{M}_{r,u}^{\beta,4})]$
2. $r_E(\mathcal{M}_{k,u}^\beta) \leq \frac{11}{6} [2^{3 \cdot k+2} - 2^{k+r} (2^r + 3 \cdot 2^k)] + [2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\beta,2}) + 2^k r_E(\mathcal{M}_{r,u}^{\beta,4})]$

Proof. By Theorem 3.4

$$\begin{aligned} r_L(\mathcal{M}_{k,u}^\beta) &\leq r_L(2^{2 \cdot k} \mathcal{M}_{k,u}^{\beta,2}) + r_L(2^k \mathcal{M}_{k,u}^{\beta,4}) \\ &\leq 2^{2 \cdot k} r_L(\mathcal{M}_{k,u}^{\beta,2}) + 2^k r_L(\mathcal{M}_{k,u}^{\beta,4}) \\ &\leq 2^{2 \cdot k} r_H(\mathcal{M}_{k,u}^{\beta,2}) + 2^k r_L(\mathcal{M}_{k,u}^{\beta,4}) \\ &\leq 2^k [2^{k-1} (2^k - 1) - 2^{r-1} (2^r - 1)] + 2^k r_L(\mathcal{M}_{r,u}^{\beta,4}) \\ &\quad + 2^k (2^k - 2^r) + 2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\beta,2}) \\ &\leq [2^{2 \cdot k-1} (3 \cdot 2^k - 1) - 2^{k+r-1} (2^{k-1} + 2^r - 1)] \\ &\quad + [2^{2 \cdot k} r_H(\mathcal{M}_{r,u}^{\beta,2}) + 2^k r_L(\mathcal{M}_{r,u}^{\beta,4})] \end{aligned}$$

□

Similar arguments holds for $r_E(\mathcal{M}_{k,u}^\beta)$.

Theorem 5.3 $r_L(\mathcal{M}_{k,u}^{\alpha\perp}) = r_L(\mathcal{M}_{k,u}^{\beta\perp}) = 2$, $r_E(\mathcal{M}_{k,u}^{\alpha\perp}) \leq 6$ and $r_E(\mathcal{M}_{k,u}^{\beta\perp}) \leq 6$

Proof. The same proof such as the theorem 4.3. □

6 $\mathbb{Z}_2\mathbb{Z}_4$ -Reed-Muller Code

In [19] and [20] the additive Reed-Muller codes over $\mathbb{Z}_2\mathbb{Z}_4$ is known by $\mathcal{ARM}(r, m)$. Just as there is only one $\mathcal{RM}(r, m)$ family in the binary case, in the $\mathbb{Z}_2\mathbb{Z}_4$ - additive case there are $\lfloor \frac{m+2}{2} \rfloor$ families for each value of m . Each one of these families will contain any of the $\lfloor \frac{m+2}{2} \rfloor$ non-isomorphic $\mathbb{Z}_2\mathbb{Z}_4$ -linear extended perfect codes which are known to exist for any m (see [8]). We will identify each family $\mathcal{ARM}_s(r, m)$ by a subindex $s \in \{0, \dots, \lfloor \frac{m}{2} \rfloor\}$. As in linear case we have the generator matrix for the code $\mathcal{ARM}(r, m)$ is given by

$$G(r, m) = \left[\begin{array}{c|c} G(r, m-1) & G(r, m-1) \\ \hline 0 & G(r-1, m-1) \end{array} \right]$$

Proposition 6.1 [20] *The additive Reed-Muller Code $\mathcal{ARM}(r, m)$ of order r has the following properties*

- 1 Minimum distance $d = 2^{m-r}$
- 2 If $k = \sum_{i=0}^r \binom{m}{i}$ then $|\mathcal{ARM}(r, m)| = 2^k$
- 3 There exists a coordinate permutation σ_r such that $\sigma_r(\mathcal{ARM}(r-1, m)) \subset \mathcal{ARM}(r, m)$, for $r > 0$

In this section we give the covering radius of first order Reed Muller code over $\mathbb{Z}_2\mathbb{Z}_4$. Let $2 \leq i \leq m-1$. Let v_i be a vector of length 2^{m-1} consisting of successive blocks of 00's and 11's each of size $2^{(m-1)-i}$ and if $\overline{11} = (11 \ 11 \ \dots \ 11) \in (\mathbb{Z}_2\mathbb{Z}_4)^{2^{m-1}}$. Let G be a $2^{m-1} \times (m-1)$ matrix given by

$$G(1, m-1) = \begin{pmatrix} 00 & 00 & \dots & 00 & 00 & 02 & 02 & \dots & 02 & 02 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 00 & 02 & \dots & 00 & 02 & 00 & 02 & \dots & 00 & 02 \\ 11 & 11 & \dots & 11 & 11 & 11 & 11 & \dots & 11 & 11 \end{pmatrix}$$

The matrix $G(1, m-1)$ is also takes the form

$$\left(\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ 0_{2^{m-2} \times (m-2)} & & & 2 \cdot S_{2,m-2}^\alpha & & \\ \hline 11 & \dots & 11 & 11 & \dots & 11 \end{array} \right),$$

where $S_{2,m-2}^\alpha$ is a binary simplex code of type α .

The code generated by $G(1, m-1)$ is called the first order Reed-Muller code over $\mathbb{Z}_2\mathbb{Z}_4$, denoted $\mathcal{RM}(1, m-1)$. It is a linear code over $\mathbb{Z}_2\mathbb{Z}_4$ [14].

Theorem 6.2 *If C is the code generated by G then $r_L(C) = r_E(C) = 2^{m-1}$.*

Proof. Let $x = 1111 \cdots 11 \in (\mathbb{Z}_2\mathbb{Z}_4)^{2^{m-1}}$, then $d_L(x, C) = d_E(x, C) = 2^{m-1}$. \square

7 The Binary Gray Images of Simplex and MacDonald Codes over $\mathbb{Z}_2\mathbb{Z}_4$

The binary version of S_k^α a simplex Codes over $\mathbb{Z}_2\mathbb{Z}_4$ is given by the following theorem.

Theorem 7.1 *Let S_k^α is a $\mathbb{Z}_2\mathbb{Z}_4$ -simplex code of type α of minimum Lee weight d_L , then $\Phi_L(S_k^\alpha)$ is a concatenation of $2^{2k} (2^k + 1)$ binary simplex code with parameters $[2^{3k} (2^k + 1); k; d_H]$.*

Proof. If Θ_k^α is generator matrix of $\mathbb{Z}_2\mathbb{Z}_4$ -simplex Codes S_k^α . Then $\Phi_L(\Theta_k^\alpha)$ is in the form

$$\Phi_L(\Theta_k^\alpha) = \left(\overbrace{m_k \mid m_k \mid \cdots \mid m_k}^{2^{2k}(2^k+1)} \right)$$

where m_k is generator matrix of binary simplex code S_k . The result follows obtained by induction on k . \square

The binary version of S_k^β a simplex Codes over $\mathbb{Z}_2\mathbb{Z}_4$ is given by the following theorem.

Theorem 7.2 *Let S_k^β is a $\mathbb{Z}_2\mathbb{Z}_4$ -simplex code of type β of minimum Lee weight d_L , then $\Phi_L(S_k^\beta)$ is a concatenation of $2^k (2^{k-1} + 1)$ binary simplex code with parameters $[2^k (2^{k-1} + 1) (2^k - 1); k; d_H]$.*

Proof. Same as the proof in theorem 7.1 □

By analogy the binary images of $\mathcal{M}_{k,u}^\alpha$ (resp., $\mathcal{M}_{k,u}^\beta$) a MacDonald and Reed-Muller Codes over $\mathbb{Z}_2\mathbb{Z}_4$ is given by the following theorem.

Theorem 7.3 *Let $\mathcal{M}_{k,u}^\alpha$ (resp., $\mathcal{M}_{k,u}^\beta$) are a $\mathbb{Z}_2\mathbb{Z}_4$ -simplex code of type α and β of minimum Lee weight d_L , then $\Phi_L(\mathcal{M}_{k,u}^\alpha)$ (resp., $\Phi_L(\mathcal{M}_{k,u}^\beta)$) is a concatenation of $2^{2(k-1)}(2^{k-1}+1)$ (resp., $2^{2(k-1)}(2^k-1)$) binary simplex code with parameters $[2^{2(k-1)}(2^{k-1}+1)(2^k-2^u); k; d_H]$ (resp., $[2^{2(k-1)}(2^k-1)(2^k-2^u); k; d_H]$).*

Proof. by Theorems 7.1 and 7.2 the results are easy. \square

Theorem 7.4 *The additive first order Reed-Muller code $\mathcal{ARM}(1, m-1)$ over $\mathbb{Z}_2\mathbb{Z}_4$ of minimum Lee weight d_L , to a binary image $\Phi_L(\mathcal{ARM}(1, m-1))$ is a code with parameters $[3 \cdot 2^{m-1}; m-1; d_H = 2^{m-2}]$*

Proof. If $G(1, m-1)$ is the matrix of the additive first order Reed-Muller code $\mathcal{ARM}(1, m-1)$ over $\mathbb{Z}_2\mathbb{Z}_4$, then the image under the Gray map is a binary code of generator matrix given by

$$\Phi_L(G(1, m-1)) = \left(\begin{array}{c|c|c} 0_{2^{m-2} \times (m-2)} & \overline{\mathcal{G}(1, m-2)} & \mathcal{G}(1, m-2) \\ \hline 11 \ \dots \ 11 & 00 \ \dots \ 00 & 11 \ \dots \ 11 \end{array} \right)$$

, where $\mathcal{G}(1, m-2)$ is a matrix of binary first order Reed-Muller code. For the minimum distance it takes two vectors u and v in $\mathcal{ARM}(1, m-1)$, we have $d_H = 2^{m-2}$.

Conclusion In this paper, we have computed bounds on the covering radius of Simplex and MacDonald codes of type α and β over $\mathbb{Z}_2\mathbb{Z}_4$ and also provided exact values in some cases. Of course, another direction and interesting research in this topic is the computed bounds on the covering radius of Simplex and MacDonald codes of type α and β over other ring.

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